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On oblique Alfvèn waves in a viscous and resistive atmosphere

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Abstract. We consider Alfvèn waves propagating obliquely in an atmosphere, subject to a uniform magnetic field of arbitrary direction, in the presence of viscous stresses and electrical resistance. This problem is fundamental to theories of atmospheric heating by dissipation of Alfvèn waves, on which there is a relatively substantial literature. The Alfvèn wave equation is deduced for an atmosphere with non-uniform diffusivities and propagation speed. The wave equation is solved exactly in the case of an isothermal atmosphere, for which the Alfvèn speed increases exponentially on twice the scale height and the dynamic viscosity increases exponentially on the scale height; the rate of ionisation is assumed uniform, leading to a constant electrical diffusivity. The exact solution includes, as particular cases, those obtained before for Alfvèn waves in an isothermal atmosphere, in the non-dissipative case with vertical (Ferraro and Plumpton) and oblique (Schwartz *et al*) magnetic field, and in the case of resistive dissipation alone (Campos). The wave fields are expressed at all altitudes in terms of hypergeometric functions, which are used to plot the amplitudes and phases for several combinations of wave frequency, horizontal wavenumber, inclination of the magnetic field to the vertical and viscous and resistive diffusivities. It is shown that, for certain ranges of values of the parameters, intense localised dissipation of waves can occur. This physical mechanism for atmospheric heating is based on (i) an exact solution of the Alfvèn wave equation, including the effects of (ii) viscous and resistive dissipation and (iii) the change of propagation speeds and damping rates with altitude; the properties of waves are compared with earlier theories, neglecting one of the effects mentioned above, viz (i) the phase mixing approximation (Heyvaerts and Priest), (ii) the resonance model (Hollweg), (iii) the *RLC* analogy (Ionson) and (iv) ray theory (Osterbrock).

1. Introduction

The theoretical prediction of the existence of hydromagnetic waves (Alfvèn 1942) was soon followed by applications to solar physics (Alfvèn 1943, 1945), including a study of atmospheric heating by wave dissipation (Alfvèn 1947). The latter points to important extensions of the non-dissipative 'Alfvèn' wave in a homogeneous medium: (i) the effect of variations of density with height in an atmosphere, in causing a non-uniform propagation speed, and thus a deformation of the sinusoidal waveform; (ii) the damping by dissipation mechanisms, viz for a transversal incompressible wave, mainly electrical resistance and viscous stresses. The purpose of the present introduction is to outline the way in which research on Alfvèn waves has evolved with regard to issues (i) and (ii) indicated above, to show that in the last 40 years, with a single exception (Campos 1983a), no exact solutions of the Alfvèn wave equation in a dissipative atmosphere have been obtained. The purpose of the present paper is to extend this solution and discuss its properties, which may have implications on other topics mentioned in the introduction, such as theories of atmospheric heating.

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The theoretical 'discovery' of hydromagnetic waves was substantiated by laboratory experiments (Lundquist 1949, Lehnert 1951) before Alfvèn waves were observed in nature, e.g. in the interplanetary medium (Belcher *et al* 1969, Belcher and Davis 1971) and in the solar wind (Burlaga and Turner 1976, Denskat and Burlaga 1977) and atmosphere (Sawyer 1974, Giovanelli and Beckers 1982). The first important theoretical extension of the Alfvèn wave was the study of coupling with compressibility, both regarding propagation (Astrom 1950, Herlofson 1950, Banos 1955) and generation (Lighthill 1960, Campos 1977, Stein 1981); the subject of magneto-acoustic waves is a well established area of magnetohydrodynamics of MHD (Alfvèn 1948, Cowling 1960, Alfvèn and Falthammar 1962, Ferraro and Plumpton 1963, Shercliff 1965, Cabannes 1970). It should be borne in mind that Alfvèn waves in atmospheres generally have propagation speeds and damping rates varying with altitude; this leads to wave equations with variable coefficients, whose properties can be quite different from the case of MHD of homogeneous media, e.g. sinusoidal solutions generally do not exist.

The study of Alfvèn waves in an inhomogeneous medium started by the consideration of refraction at interfaces (Ferraro 1954, Stein 1971), and proceeded to media with continuously varying properties (Hide 1955), e.g. due to the presence of gravity (Howe 1969). Although it was predicted fairly early that the decay of mass density with altitude would significantly affect the properties of Alfvèn waves (Wallen 1944), by making the propagation speed non-uniform, the first exact solution, for an isothermal atmosphere, was obtained much later (Ferraro and Plumpton 1958). This solution can be arrived at in several different ways (Campos 1983b) and has been used many times, e.g. in solar atmospheric models using multiple isothermal layers (Hollweg 1972, 1978, 1981, 1984a) or an isothermal layer with a homogeneous medium on top (Leroy 1980, 1981, Schwartz *et al* 1984). Some of these applications of Alfvèn waves in atmospheres have tried to address the question of atmospheric heating, although the wave fields are calculated neglecting dissipation—an obvious weak point.

The underlying assumption of atmospheric models using multiple isothermal layers, namely that the discontinuities of temperature represent wave reflection and transmission as well as a continuous temperature profile, is yet to be checked. The approach of considering an isothermal atmosphere with a homogeneous medium on top has the attraction of allowing the identification of ordinary magneto-acoustic modes in the upper layer; it faces the more serious objection that hydrostatic equilibrium is not satisfied by an 'infinite' homogeneous layer on top of an isothermal atmosphere. The case of Alfvèn waves in a polytropic atmosphere, with a linear temperature gradient, has also been considered (Zhugzhda 1971, Parker 1984) and applied to the solar atmosphere (Thomas 1978, Zhugzhda and Locans 1982); this model avoids discontinuities of temperature but is restricted to layers of finite thickness, otherwise the temperature would diverge. For Alfvèn waves in dissipative atmospheres, it may be more important (Campos 1983c) not to neglect damping mechanisms, such as fluid viscosity and electrical resistance, than to go into much detail (Campos 1983d) about mean-state temperature profiles, which can be modified by wave dissipation.

The knowledge of Alfvèn waves has progressed in several directions, including analytical (Barnes and Hollweg 1974, Lacombe and Mangeney 1980) and numerical (Hollweg *et al* 1982, Mariska and Hollweg 1985) studies of non-linear waves, the continuous spectrum (Kieras and Tataronis 1982, Connor *et al* 1983, Mahajan and Ross 1983, Goedbloed 1984) and surface modes (Roberts 1981, Edwin and Roberts 1982, Narayan and Somasundaram 1985), parametric generation (Petrukhin and Fainshtein 1984) and the effects of displacement currents (Leroy 1983) and resonant modes

(Campos 1986a). Some backward steps have also been taken, at least from the point of view of analytical modelling, e.g. the replacement of Alfvèn waves in an atmosphere by an *RLC* circuit 'analogy' (Ionson 1982, 1984) that neglects the variation of Alfvèn speed with altitude, taken into account by most authors for over a quarter of a century (since Ferraro and Plumpton 1958). The claim that the *RLC* analogy can model the heating of coronal loops (Ionson 1985, Kuperus *et al* 1981) has been matched by a rather different model (Hollweg and Sterling 1984). The reason dissimilar models give similar results, in agreement with observation, may have more to do with the assumed input energy spectrum at the photosphere than with the properties of the waves.

Whichever view is taken on the subject of atmospheric heating by waves, e.g. in the solar (Osterbrock 1961, Campos 1984, Hollweg 1984c) or stellar (Ulmschneider 1979) case, it is clear that a fundamental input is the calculation of the wave field, taking into account the variation of propagation speed and dissipation rates with altitude. The exact solution of the Alfvèn wave equation with damping in an atmosphere has been obtained (Campos 1983a) in the isothermal case for propagation along a vertical magnetic field; in this case the Alfvèn speed increases exponentially with altitude divided by twice the scale height and the magnetic diffusivity was assumed to be constant. In the present paper the problem is extended in three ways: (i) the external magnetic field may have an arbitrary inclination to the vertical; (ii) the waves may be oblique, i.e. have a non-zero horizontal wavenumber, besides a generally non-sinusoidal dependence on altitude; (iii) besides electrical resistance, the other form of dissipation of linear Alfvèn waves, namely viscosity, is also included, on the assumption of constant static viscosity, and hence dynamic viscosity growing exponentially with altitude divided by the scale height. The exact solution includes, as particular cases, earlier results concerning non-dissipative waves in vertical (Ferraro and Plumpton 1958) and oblique (Schwartz *et al* 1984) magnetic fields and dissipation by electrical resistance alone (Campos 1983a).

The Alfvèn wave equation in a viscous and resistive atmosphere is valid in non-isothermal conditions and for uniform resistive diffusivity and viscosity varying in an arbitrary manner with altitude (Campos 1987). The equation has been considered in the context of the 'phase mixing approximation' (Sakurai and Granik 1984, Steinholfson 1985), which is based on an assumed form of the solution of the wave equation (Heyvaerts and Priest 1983, Nocera *et al* 1984). The assumption is that a wavenumber exists in the direction of propagation, which is transverse to the direction of non-uniformity of the medium, and depends on the latter. The phase mixing theory starts from an incorrect form of the dissipative Alfvèn wave equation (see § 2) and our exact solution does not exhibit large space shifts (see § 6) as implied by the phase mixing approximation. The exact solution has a critical level, of transition layer type (Campos 1988), and the wave fields are calculated at all altitudes, i.e. below, above and at the critical level. Thus we can plot the wave amplitude and phase, as a function of altitude, for a range of values of the wave frequency, horizontal wavenumber, inclination of the magnetic field to the vertical and viscous and resistive damping rates, showing the conditions in which localised intense dissipation can occur.

2. Wave equations for velocity and magnetic field perturbations

Dissipative Alfvèn waves are considered in the literature (Cowling 1960, Moffatt 1976, Ionson 1982) in the case of uniform wave speed and viscous and resistive diffusivities;

the case of non-uniform wave speed has also been discussed (Heyvaerts and Priest 1983), but the equation derived is in error. Since we need the viscous and resistive Alfvèn wave equation, in the case of wave speed and diffusivities varying in one dimension, e.g. altitude, we give a short derivation which also indicates the terms overlooked in the literature (Heyvaerts and Priest 1983). We consider a mean state of magnetohydrostatic equilibrium, with mass density $\rho(z)$ depending only on altitude; this is consistent with (a) a non-uniform horizontal magnetic field $\boldsymbol{B}=B(z)\boldsymbol{e}_x$, which corresponds to the case of dissipative Alfvèn waves in a magnetic slab, or flux tube, and will be discussed in a separate paper; (b) an oblique uniform magnetic field $\boldsymbol{B}=B(n\boldsymbol{e}_z+m\boldsymbol{e}_x)$, where $n\equiv\cos\theta$, $m=\sin\theta$ and θ is the angle of the magnetic field to the vertical. Case (a) applies to dissipative Alfvèn waves in the low solar atmosphere (photosphere and low chromosphere), where the magnetic field is organised into flux tubes (Roberts 1981); case (b) applies after the flux tubes have merged into a nearly uniform magnetic field (Gabriel 1976), in the high chromosphere and corona, and will be considered in the following. The velocity $\boldsymbol{v}$ corresponds to a wave perturbation over the mean atmospheric state of rest and the total magnetic field $\boldsymbol{H}$ to the superposition of a perturbation $B\boldsymbol{h}$ on the external magnetic field:

$$\boldsymbol{v}(\boldsymbol{x},t)=v(x,z,t)\boldsymbol{e}_y \qquad (1a)$$

$$\boldsymbol{H}(\boldsymbol{x},t)=B[n\boldsymbol{e}_z+m\boldsymbol{e}_x+h(x,z,t)\boldsymbol{e}_y]. \qquad (1b)$$

The wave is assumed to be transversal, i.e. the velocity $\boldsymbol{v}$ and magnetic field $\boldsymbol{h}$ perturbations are orthogonal to the plane (x, z) of gravity $\boldsymbol{g}$ (or stratification z) and the exernal magnetic field, and depend only on the variables in that plane (see figure 1).

The equations of induction $(2a)$ and momentum $(2b)$ are

$$\partial h/\partial t-\partial v/\partial l=\chi\nabla^2 h-\chi' h' \qquad (2a)$$

$$\partial v/\partial t-A^2\,\partial h/\partial l=\eta\nabla^2 v \qquad (2b)$$

where the prime denotes derivative with respect to altitude $x'\equiv dx/dz, \nabla^2$ is the two-dimensional Laplacian $(3a)$ in the (x, z) plane and $\partial/\partial l$ is the derivative $(3b)$ along magnetic field lines:

$$\nabla^2\equiv\partial^2/\nabla x^2+\partial^2/\partial z^2 \qquad (3a)$$

$$\partial/\partial l\equiv n\,\partial/\partial z+m\,\partial/\partial x=\cos\theta\,\partial/\partial z+\sin\theta\,\partial/\partial x \qquad (3b)$$

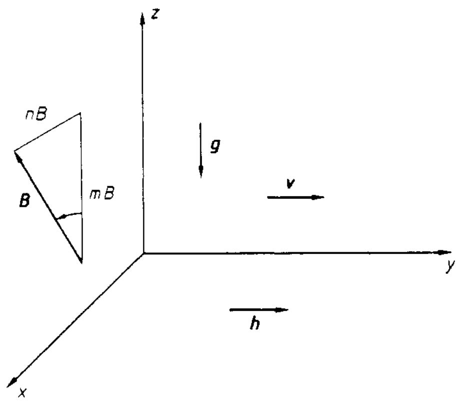

Figure 1. Model geometry. Vertical z direction opposite to gravity $\boldsymbol{g}$, uniform magnetic field $\boldsymbol{B}$ in the $x0z$ plane making an angle θ with the vertical, and transverse velocity $\boldsymbol{v}$ and magnetic field $\boldsymbol{h}$ perturbations in the y direction.

and the resistive χ and viscous η diffusivities may depend on altitude z, as well as the Alfvèn speed:

$$\{A(z)\}^2 = \mu B^2/4\pi\rho(z) \tag{4a}$$

$$\chi(z) = c^2/4\pi\mu\sigma(z). \tag{4b}$$

In $(4a, b)$ μ denotes the magnetic permeability, c is the speed of light *in vacuo* and $\sigma(z)$ is the ohmic electrical conductivity. In the case of uniform electrical diffusivity $\chi' = 0$ and variable viscosity $\eta(z)$ and Alfvèn speed $A(z)$, we may eliminate h in $(2a, b)$, i.e. solve for the velocity perturbation:

$$\chi' = 0 \qquad (\partial/\partial t - \chi\nabla^2)A^{-2}(\partial/\partial t - \eta\nabla^2)v = (\partial/\partial t - \chi\nabla^2)\,\partial h/\partial l = \partial^2 v/\partial l^2. \tag{5a}$$

In the case of uniform viscous diffusivity $\eta'' = 0$ and variable resistive diffusivity $x(z)$ and Alfvèn speed $A(z)$, we may eliminate v in $(2a, b)$, i.e. solve for the magnetic field perturbation:

$$\chi' = 0 = \nu' \qquad (\partial/\partial t - \eta\nabla^2)(\partial/\partial t - \chi\nabla^2)h = (\partial/\partial t - \eta\nabla^2)\,\partial v/\partial l = \partial/\partial l A^2\,\partial h/\partial l. \tag{5b}$$

Thus we obtain the Alfvèn wave equations, with viscous and resistive dissipation, for the velocity $(6a)$ and magnetic field $(6b)$ perturbations:

$$\partial^2 v/\partial t^2 - A^2\,\partial^2 v/\partial l = \eta\nabla^2(\partial v/\partial t) + A^2\chi\nabla^2(A^{-2}\,\partial v/\partial t) - A^2\chi\nabla^2(A^{-2}\eta\nabla^2 v) \tag{6a}$$

$$\partial^2 h/\partial t^2 - \partial/\partial l(A^2\,\partial h/\partial l) = (\eta+\chi)\nabla^2(\partial h/\partial t) - \chi\nabla^2(\nabla^2 h). \tag{6b}$$

In the case of uniform diffusivities and Alfvèn speed, the two wave equations $(6a, b)$ coincide:

$$\chi' = \eta' = A' = 0 \qquad [\partial^2/\partial t^2 - A^2\,\partial^2/\partial l^2 - (\nu+\chi)\nabla^2\,\partial/\partial t + \nu\chi\nabla^2] \qquad v, h = 0. \tag{7}$$

The wave equation (7) is stated (Heyvaerts and Priest 1983) to hold for the velocity in the case of uniform diffusivities $\chi' = 0 = \eta'$ and non-uniform Alfvèn speed $A(z)$; this is erroneous, since $A^2\nabla^2(A^{-2}\partial v/\partial t) \neq \nabla^2(\partial v/\partial t)$ and $(6a)$ does not coincide with (7). Since Alfvèn waves are transversal, i.e. incompressible, they are not affected by thermal conduction or radiation, unless they couple non-linearly to compressive modes. Thus linear Alfvèn waves are damped by viscosity and electrical resistance only and $(6a, b)$ are the general forms of the dissipative wave equations.

Since the atmospheric mean state is steady and horizontally homogeneous, we may use a Fourier decomposition in time t and horizontal coordinate x:

$$v(x, z, t) = \int\int_{-\infty}^{+\infty} V(z; k, \omega)\exp[\mathrm{i}(kx - \omega t)]\,\mathrm{d}k\,\mathrm{d}\omega \tag{8}$$

where $V(z; k, \omega)$ is the velocity perturbation spectrum, for a wave of frequency ω and horizontal wavenumber k, at altitude z. The dissipative wave equation for the velocity $(6a)$ is of fourth order, but in the case of weak damping, the product $\chi\eta$ is neglected (Heyvaerts and Priest 1983) and, omitting the last term, we obtain a second-order equation for the velocity perturbation spectrum:

$$\begin{aligned}[n^2A^2 - \mathrm{i}\omega(\chi+\eta)]V'' + 2\mathrm{i}(kmnA^2 + 2\omega\chi A'/A)V' \\ + [\omega^2 - k^2m^2A^2 + \mathrm{i}\omega k^2(\eta+\chi) + 2\mathrm{i}\omega\chi(A''/A - 3A'^2/A^2)]V = 0\end{aligned} \tag{9}$$

where the prime denotes, as before ($(2a)$, $(5a, b)$ and (7)) the derivative with regard to altitude $V' \equiv \mathrm{d}V/\mathrm{d}z$. The coefficients of (9) depend on the atmospheric mean state and may be specified as follows.

(i) We consider an isothermal layer, for which the density $\rho(z)$ decays exponentially depending on the scale height L (10c) and for a uniform magnetic field B, the Alfvèn speed (4a) increases at twice that scale (10a) from an initial value (10b) at altitude $z=0$:

$$A(z)=a\exp(z/2L) \tag{10a}$$

$$a^2\equiv \mu B^2/4\pi\rho(0) \tag{10b}$$

$$L\equiv RT/g. \tag{10c}$$

(ii) The kinematic viscosity $\bar{\eta}$ depends mainly on temperature and is constant in an isothermal layer, so that the dynamic viscosity of viscous diffusivity, $\eta(z)=\bar{\eta}/\rho_0(z)$, varies inversely with density, i.e. grows exponentially (11a) depending on the scale height (10c):

$$\eta(z)=\eta_0\exp(z/L) \tag{11a}$$

$$\chi(z)=\chi_0. \tag{11b}$$

This states that the electrical diffusivity $\chi(z)$, which depends mainly on temperature and rate of ionisation, is constant (11b) for an isothermal layer with constant rate of ionisation. In this case ((10a) and (11a, b)), the wave equation (9) for the velocity perturbation:

$$\{n^2-\mathrm{i}[\delta+\varepsilon\exp(-z/L)]\}L^2V''+2\mathrm{i}[Kmn+\varepsilon\exp(-z/L)]LV' \\ +\{K^2(\mathrm{i}\delta-m^2)+[\Omega^2+\mathrm{i}\varepsilon(K^2-1)]\exp(-z/L)\}V=0 \tag{12a}$$

involves four dimensionless parameters, namely the frequency Ω, horizontal compactness K and viscous δ and resistive ε damping:

$$\Omega\equiv\omega L/a \tag{12b}$$

$$K\equiv kL \tag{12c}$$

$$\delta\equiv\eta_0\omega/a^2 \tag{12d}$$

$$\varepsilon\equiv\chi_0\omega/a^2. \tag{12e}$$

We perform in (12a) the changes of independent (13a) and dependent (13b) variables:

$$\rho=[\mathrm{i}\varepsilon/(n^2-\mathrm{i}\delta)]\exp(-z/L) \tag{13a}$$

$$V(z;k,\omega)=\rho^\nu\psi(\rho) \tag{13b}$$

where ν is a constant, which specifies the asymptotic behaviour at high altitude $z\to\infty$, since $\rho\to 0$ and $V(z;k,\omega)\sim\exp(-\nu z/L)$ for $\psi(0)$ finite. We choose ν so that the coefficient of ψ does not depend on ρ, i.e. ν is a root of

$$(n^2-\mathrm{i}\delta)\nu^2-2\mathrm{i}Kmn\nu-K^2(m^2-\mathrm{i}\delta)=0. \tag{14a}$$

It follows from (14a) that the differential equation for ψ, obtained by substituting (13a, b) into (12), can be divided throughout by ρ, leading to

$$(1-\rho)\rho\psi''+[1+2\nu-2\mathrm{i}Kmn/(n^2-\mathrm{i}\delta)-(3+2\nu)\rho]\psi' \\ -(\nu^2+2\nu+1-K^2+\mathrm{i}\Omega^2/\varepsilon)\psi=0 \tag{14b}$$

where the prime denotes $\psi'\equiv \mathrm{d}\psi/\mathrm{d}\rho$, the derivative with regard to ρ.

3. Exact general solution and particular cases

The differential equation (14*b*) is of the hypergeometric (Kamke 1944) type:

$$(1-\rho)\rho\psi''+[\gamma-(\alpha+\beta+1)\rho]\psi'-\alpha\beta\psi=0 \tag{15}$$

with parameters α, β and γ satisfying

$$\gamma=1+2\nu-2\mathrm{i}Kmn/(n^2-\mathrm{i}\delta) \tag{16a}$$

$$\alpha+\beta=2(1+\nu) \tag{16b}$$

$$\alpha\beta=(\nu+1)^2-K^2+\mathrm{i}\Omega^2/\varepsilon. \tag{16c}$$

The solution at high altitude, i.e. for large z and small ρ in (13*a*), is given (Forsyth 1929) by

$$\psi(\rho)=C_1F(\alpha,\beta;\gamma;\rho)+C_2\rho^{1-\gamma}F(1+\alpha-\gamma,1+\beta-\gamma;2-\gamma;\rho) \tag{17}$$

as a linear combination, with constant coefficients, C_1, C_2, of hypergeometric functions (Erdelyi 1953) of Gaussian type $F\equiv {}_2F_1$. Once ν is found as a root of (14*a*), the parameter γ is given explicitly by (16*a*) and α, β by the solution of (16*b*, *c*), viz

$$\alpha=1+\nu+(K^2-\mathrm{i}\Omega^2/\varepsilon)^{1/2} \tag{18a}$$

$$\beta=1+\nu-(K^2-\mathrm{i}\Omega^2/\varepsilon)^{1/2}. \tag{18b}$$

The wave field (13*b*) is specified by

$$V(z;k;\omega)=C_1\rho^{\nu}F(\alpha,\beta;\gamma;\rho)+C_2\rho^{1+\nu-\gamma}F(1+\alpha-\gamma,1+\beta-\gamma;2-\gamma;\rho) \tag{19}$$

in terms of Gaussian hypergeometric functions of the first kind, provided that $\gamma\neq 1$ (Poole 1937).

The two particular integrals in (19) coincide for $\gamma=1$, so that the two constants of integration would coalesce into one $C_1+C_2\equiv C_0$, i.e. (19) is not the general integral in the case $\gamma=1$ and hypergeometric functions of the second kind G must be introduced (Caratheodory 1935). The condition $\gamma=1$ in (16*a*) implies (20*a*):

$$\nu=\mathrm{i}Kmn/(n^2-\mathrm{i}\delta) \tag{20a}$$

$$K^2\delta(\mathrm{i}+\delta)=0 \tag{20b}$$

where the second condition (20*b*) results from the substitution of (20*a*) into (14*a*). Thus, we have two cases. (i) If $K=0$, i.e. for vertical waves, with viscous damping $\delta\neq 0$:

$$K=0 \tag{21a}$$

$$\nu_0=0 \tag{21b}$$

$$\gamma_0=1 \tag{21c}$$

$$\alpha_0,\beta_0=1\pm(\mathrm{i}-1)\Omega/\sqrt{2\varepsilon}. \tag{21d}$$

The wave field is a linear combination of hypergeometric functions of the first (F) and second (G) kinds:

$$V(z;0;\omega)=\psi(\rho)=C_1F(\alpha_0,\beta_0;1;\rho)+C_2G(\alpha_0,\beta_0:\gamma_0;\rho). \tag{22}$$

(ii) If $\delta=0$, i.e. in the absence of viscous dissipation but for oblique waves $K\neq 0$, we have the parameters

$$\delta=0 \tag{23a}$$

$$\nu_1=\mathrm{i}Km/n \tag{23b}$$

$$\gamma_1=1 \tag{23c}$$

$$\alpha_1,\beta_1=1+\mathrm{i}Km/n\pm(K^2-\mathrm{i}\Omega^2/\varepsilon)^{1/2} \tag{23d}$$

which replace (21a–d) in the wave field:

$$V(z;k;\omega)=\rho^{\nu}[C_1F(\alpha_1,\beta_1;1;\rho)+C_2G(\alpha_1,\beta_1;1;\rho)] \tag{24}$$

which is of the degenerate type, as (22).

The Alfvèn wave equation in a dissipative atmosphere has been solved exactly previously (Campos 1983a) in the case of vertical waves $K=0$, no viscous dissipation $\delta=0$ and vertical magnetic field $n=1$, $m=0$; in this case, electrical resistance remains as the sole dissipation mechanism, i.e. the solution is (21c,d) and (22) with $\rho=\mathrm{i}\varepsilon\exp(-z/L)$ instead of (13a). A further restriction, i.e. the neglect of dissipation, $\varepsilon=0$, leads to the often-quoted solution of the Alfvèn wave equation in an isothermal atmosphere (Ferraro and Plumpton 1958). If we take the non-dissipative limit $\varepsilon\to 0$ in the particular integrals (22):

$$\begin{aligned}\lim_{\varepsilon\to 0} F,G[\Omega(-\mathrm{i}/\varepsilon)^{1/2}, -\Omega(-\mathrm{i}/\varepsilon)^{1/2}; 1; \mathrm{i}\varepsilon\exp(-z/L)]\\ =J_0, Y_0\{[2(-\mathrm{i}\Omega^2/\varepsilon)(\mathrm{i}\varepsilon\exp(-z/L))]^{1/2}\}\\ =J_0, Y_0[2\Omega\exp(-z/2L)]\end{aligned} \tag{25a}$$

the hypergeometric functions of the first (F) and second (G) kinds reduce (Watson 1944), respectively, to Bessel (J) and Neumann (Y) functions. The original non-dissipative solution (Ferraro and Plumpton 1958) assumes a vertical magnetic field and has been extended to oblique fields (Schwartz *et al* 1984); it is clear from (24) that the effect of an oblique magnetic field $n\neq 1$, $m\neq 0$ in (1b) and (3b) is to multiply the wave field (25a) by a factor

$$\rho^{\nu_1}\sim\exp(-\nu_1 z/L)=\exp(-\mathrm{i}Kmz/nL)=\exp(-\mathrm{i}kz\tan\theta) \tag{25b}$$

where θ is the angle of inclination of the magnetic field to the vertical. Thus the results obtained here apply to vertical or oblique waves, in vertical or oblique magnetic fields, with viscous and/or resistive dissipation and generalise several previous exact solutions (Ferraro and Plumpton 1958, Campos 1983a, Schwartz *et al* 1984).

4. Transition layer between initial and asymptotic wave fields

We proceed to analyse, in more detail, the general case of oblique waves of arbitrary frequency in a viscous and resistive atmosphere with magnetic field at arbitrary inclination; the parameter ν is a root of (14a), viz

$$(n^2-\mathrm{i}\delta)(\nu, 1+\nu-\gamma)=\mathrm{i}K[mn\pm(\mathrm{i}\delta+\delta^2)^{1/2}, mn\mp(\mathrm{i}\delta+\delta^2)^{1/2}] \tag{26}$$

so that exchanging the sign before the first square root in ν (26) involves, by (16a), the reverse change of sign in $1+\nu-\gamma$ (the second square root in (26)) i.e. the two particular integrals in (19) are interchanged and the general integral remains the same for arbitrary constants of integration C_1, C_2. For definiteness, we pick the lower sign in both expressions and find that, for $\delta^2 \ll 1$,

$$\rho^{\nu}, \rho^{1+\nu-\gamma} \sim \exp\{-\mathrm{i}kz[mn \mp (\delta/2)^{1/2}]n^2/(n^4+\delta^2)\}$$
$$\times \exp\{kz[mn\delta \mp n^2(\delta/2)^{1/2}]/(n^4+\delta^2)\} \qquad (27)$$

and apply the 'damping condition' (Campos 1983a) that viscosity should reduce the wave amplitude. Two cases arise.

(i) For a wave propagating in the positive x direction of the horizontal magnetic field $k>0$, we must pick the $-$ sign, i.e. select the first particular integral in (19), by setting $C_2=0$ and retaining the first term, with ν, γ given by

$$(n^2-\mathrm{i}\delta)(\nu, \gamma) = \mathrm{i}K[mn-(\mathrm{i}\delta+\delta^2)^{1/2}, n^2-\mathrm{i}\delta-2(\mathrm{i}\delta+\delta^2)^{1/2}] \qquad (28)$$

and α, β by (18*a*, *b*).

(ii) For a wave propagating in the direction opposite to the magnetic field $k<0$, we must pick the + sign in (27), i.e. select the second particular integral in (19), by setting $C_1=0$ and retaining the second term, which is equivalent to the substitutions:

$$\nu, \alpha, \beta, \gamma \to \nu+1-\gamma, \alpha+1-\gamma, \beta+1-\gamma, 2-\gamma \qquad (29)$$

where α, β are not affected by the sign of k except through ν.

The solution (19) applies for $|\rho|<1$ in (13*a*), i.e. in the high-altitude range $z>z_*$, where z_* is the altitude of the critical level, specified by $|\rho(z_*)|=1$, viz

$$|\rho(z_*)|=1 \qquad z_* = L[\log\varepsilon - \tfrac{1}{2}\log(n^4+\delta^2)]. \qquad (30)$$

The critical level is *not* of type 1, i.e. it is not a singular layer, since the coefficient $1-\rho$ of the highest-order derivative in (14*b*) vanishes for $\rho(\bar{z})=1$ at the 'altitude' $\bar{z}$, which is complex:

$$\rho(\bar{z})=1 \qquad \bar{z} = L\log[\mathrm{i}\varepsilon/(n^2-\mathrm{i}\delta)] \neq z_* \qquad (31)$$

and thus distinct from (30). We can check, substituting (30) into (13*a*):

$$\rho_* \equiv \rho(z_*) = (n^4+\delta^2)^{1/2}/(\mathrm{i}n^2-\delta) \equiv \exp(\mathrm{i}\xi) \qquad (32a)$$

$$\xi \equiv \arg(\rho_*) = \tan^{-1}(-\delta/n^2) \qquad (32b)$$

that ρ_* is distinct from unity, i.e. ρ_* lies on the unit circle $|\rho_*|=1$ in the ρ plane, but not on the positive real half-axis, i.e. $\rho_* \neq 1$ in (32*a*) or $\xi \neq 0$ in (32*b*). The implication of (31) is that $1-\rho$ is non-zero for all real altitudes $0<z<\infty$, and thus the wave field,obtained as a solution of (14*b*), is finite everywhere, including at the critical level $z=z_*$. This will be confirmed in the following, viz by calculating analytically the wave field at the critical level (see formula (39)) and also by plotting the wave field through the critical level (in figures 2–6). This kind of critical level, which is not a singularity at real altitude, is sometimes considered as an 'artefact' of the change of variable (13*a*) used to transform the wave equation (12) into a solvable form (14*b*), viz the hypergeometric type (15) and (16*a*, *b*, *c*). A closer analysis of the non-singular type of critical level shows (Campos 1987b) that it can have a definite physical meaning in the context of a classification of critical levels (Campos 1987a), of which the familiar singular layer is type I. In the present case we have a critical level of type II, i.e. a transition layer, since (i) in the low-altitude range $0>z>z_*$, below the critical level, propagation dominates dissipation and the wave field is specified, for $|\rho(z)|>1$, by a hypergeometric function of variable $1/\rho$, such that $|1/\rho|<1$ in formula (33); (ii) in the high-altitude

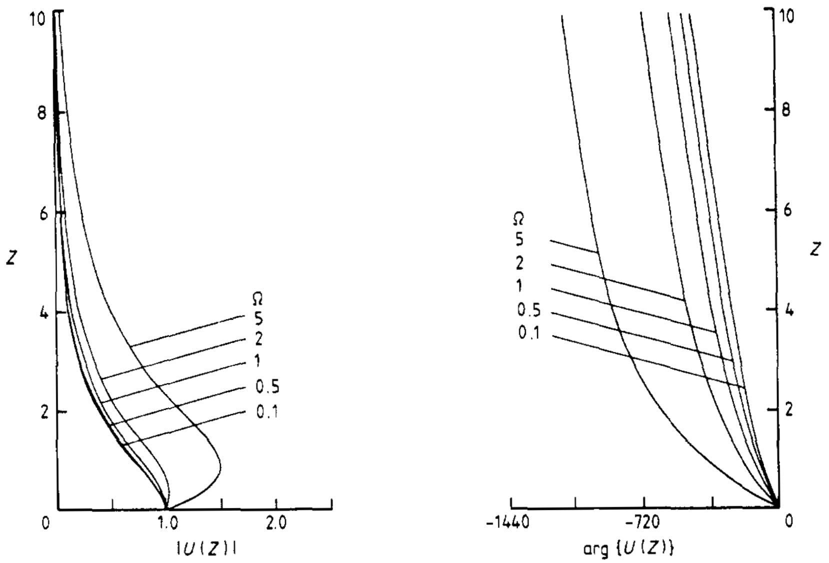

Figure 2. Ratio of amplitudes (LHS) and phase difference (RHS) between altitudes 0 and z as a function of altitude made dimensionless by dividing by the scale height L, $Z = z/L$. Effect of changing dimensionless frequency $\Omega \equiv \omega L/c$. Fixed $k = 1$, $\theta = 45°$, $\delta = 0.1 = \varepsilon$.

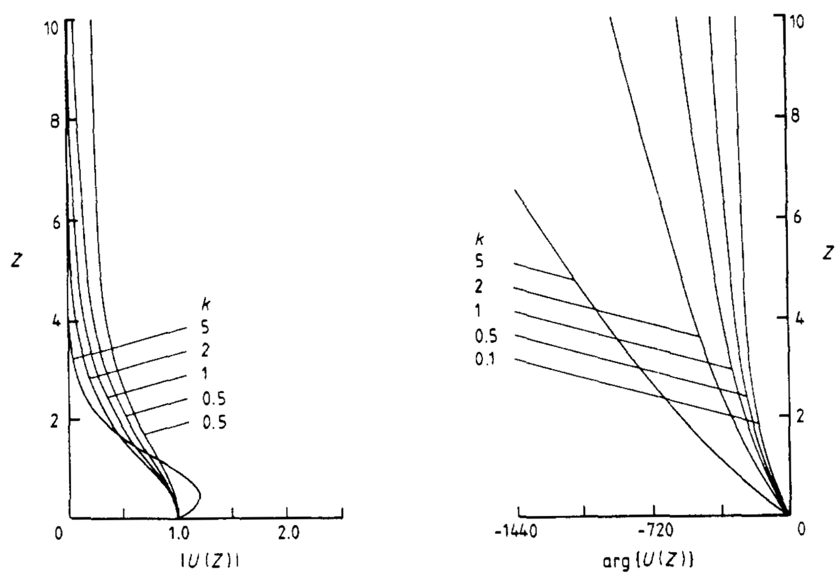

Figure 3. As for figure 2. Effect of changing horizontal compactness $K \equiv kL$. Fixed $\Omega = 1$, $\theta = 45°$, $\delta = 0.1 = \varepsilon$.

range $z_* < z < \infty$, above the critical level, dissipation dominates propagation and the wave field is specified, for $|\rho| < 1$, by hypergeometric functions of variable ρ, viz (19); (iii) at the transition layer $z = z_*$ between the low- and high-altitude regions, there is a smooth matching of the propagation-dominated regime below to the dissipation-dominated regime above, as will be shown by calculating the wave fields in the entire altitude range $0 < z < \infty$.

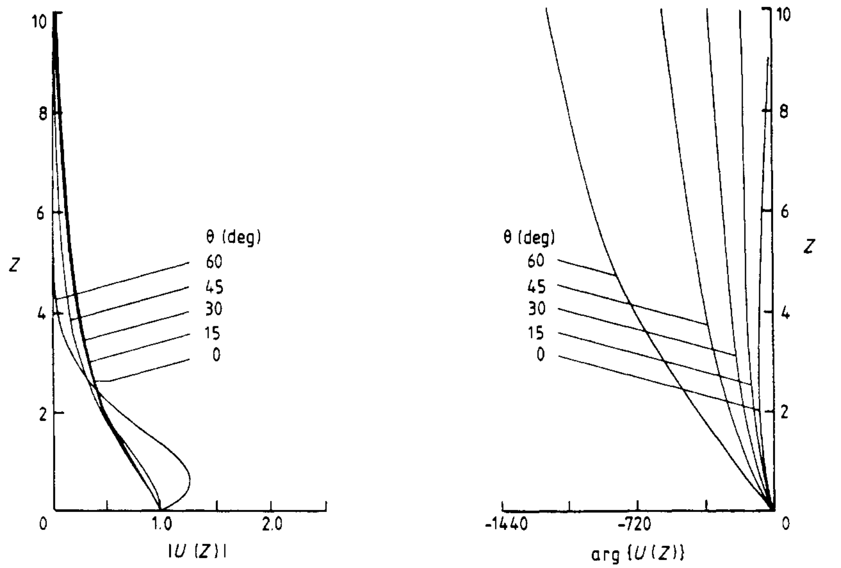

Figure 4. As for figure 2. Effect of changing the inclination θ of the magnetic field to the vertical. Fixed $\Omega = 1 = k$, $\delta = 0.1 = \varepsilon$.

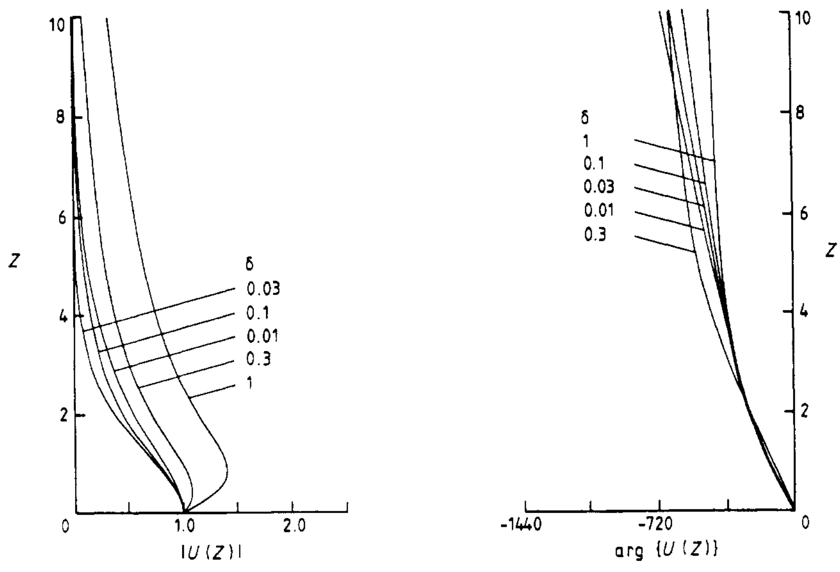

Figure 5. As for figure 2. Effect of changing viscous damping $\delta \equiv \nu_0\omega/a^2$. Fixed $\Omega = 1 = k$, $\theta = 45°$, $\varepsilon = 0.1$.

Above the critical level $\infty > z > z_*$ the wave field is specified by (19). Below the critical level $|\rho| > 1$ and thus we should use the solution of the hypergeometric equation in terms (Ince 1926) of the variable $1/\rho$, viz

$$V(z;\, k;\, \omega) = \rho^{\nu}[C_\alpha(-\rho)^{-\alpha}F(\alpha, 1+\alpha-\gamma;\, 1+\alpha-\beta;\, 1/\rho) + \text{interchange } (\alpha, \beta)] \qquad (33)$$

specifies the wave field in the altitude range $0 \leqslant z < z_*$. The relation (Abramowitz and Stegun 1965) between the hypergeometric function of variables ρ and $1/\rho$ allows the

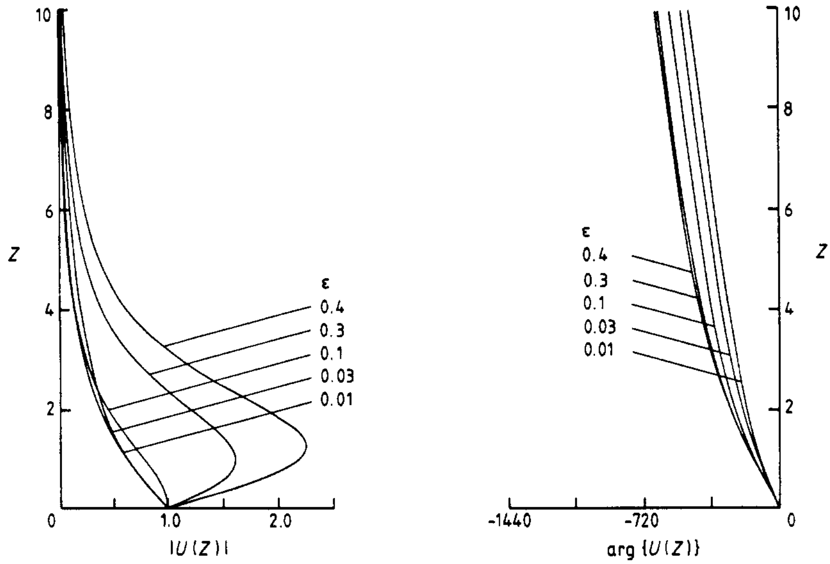

Figure 6. As for figure 2. Effect of changing resistive damping $\varepsilon \equiv \chi_0\omega/a^2$. Fixed $\Omega = 1 = k$, $\theta = 45°$, $\delta = 0.1$.

arbitrary constants C_α, C_β in (33) to be expressed:

$$C_\alpha = \Gamma(\beta-\alpha)[C_1\Gamma(\gamma)/\Gamma(\beta)\Gamma(\gamma-\alpha) + C_2\Gamma(2-\gamma)/\Gamma(1-\alpha)\Gamma(1+\beta-\gamma)] \tag{34a}$$

$$C_\beta = \text{interchange } (\alpha, \beta) \tag{34b}$$

in terms of C_1, C_2, appearing in the high-altitude solution (19). If we use the transformation formula (Morse and Feshbach 1953) between hypergeometric functions of variable $Y \equiv 1/\rho$ and $Y/(Y-1) = 1/(1-\rho)$, viz

$$F(\alpha, 1+\alpha-\gamma; 1+\alpha-\beta; 1/\rho) = (1-1/\rho)^{-\alpha} F(\alpha, \gamma-\beta; 1+\alpha-\beta; 1/(1-\rho)) \tag{35}$$

in (33), we obtain an expression for the wave field:

$$V(z; k, \omega) = \rho^\nu[C_\alpha(1-\rho)^{-\alpha}F(\alpha, \gamma-\beta; 1+\alpha-\beta; 1/(1-\rho)) + \text{interchange } (\alpha, \beta)] \tag{36}$$

valid at all altitudes $0 < z < \infty$, because $|1/(1-\rho)| < 1$, viz from (13*a*) it follows that

$$|1-\rho|^2 = [1 + \varepsilon\delta(n^4+\delta^2)^{-1}\exp(-z/L)]^2 + n^4\varepsilon^2(n^4+\delta^2)^{-2}\exp(-2z/L) > 1. \tag{37}$$

Thus the solution (36) coincides with (19) in the high $z > z_*$, and with (33) in the low $z < z_*$ altitude ranges, respectively above and below the critical level (30). Both solutions (19) and (33) diverge on the circle of convergence $|\rho| = 1$, because (Bromwich 1926)

$$\mathrm{Re}(\gamma - \alpha - \beta) = -1 - 2Kmn\delta/(n^4+\delta^2) < -1 \tag{38}$$

by (16*a*, *b*). However, the solution (36) applies at the critical level (30), where the variable ρ (13*a*) takes the value ρ_* (32*a*), with $\xi \neq 0$ in (32*b*). Thus (36) with (32*a*) specifies the wave field at the critical level:

$$V_*(k, \omega) \equiv V(z_*; k, \omega)$$
$$= \exp(i\nu\xi)\{C_\alpha[1-\exp(i\xi)]^{-\alpha}F(\alpha, \gamma-\beta; 1+\alpha-\beta; 1)/[1-\exp(i\xi)]$$
$$+\text{interchange } (\alpha, \beta)\} \qquad (39)$$

showing that the amplitude $|V_*|$ and phase $\arg(V_*)$ are finite there.

5. Effects of frequency, wavenumber, inclination and damping

For the purposes of computation and plotting of waveforms, it is most convenient to use the solution (36) valid at all altitudes $0<z<\infty$ in the dimensionless form:

$$U(Z) \equiv V(z; k, \omega)/V(0; k, \omega) \qquad (40a)$$

$$Z \equiv z/L \qquad (40b)$$

obtained by dividing the velocity perturbation spectrum at altitude z (8) by its initial value at altitude $z=0$ (40*a*), and measuring (40*b*) altitude on the scale height L (10*c*). We choose for representation the first term of (36), viz

$$U(Z) = \exp(-\nu Z)[(1-X_0)/(1-X)]^\alpha[F(1/(1-X))/F(1/(1-X_0))] \qquad (41a)$$

$$F(Y) \equiv F(\alpha, \gamma-\beta; 1+\alpha-\beta; Y) \qquad (41b)$$

which involves the (Whittaker and Watson 1927) hypergeometric function (41*b*), with parameters determined by (18*a*, *b*) and (28*a*, *b*) and variables

$$X_0 = i\varepsilon/(n^2 - i\delta) \qquad (42a)$$

$$X = X_0\exp(-Z) \qquad (42b)$$

corresponding to (13*a*) at altitude, respectively zero (42*a*) and z (42*b*).

The method of computation is as follows.

(i) We start with given values of the five dimensionless parameters, namely frequency Ω (12*b*), horizontal compactness K (12*c*), inclination θ of the magnetic field to the vertical ($n=\cos\theta$, $m=\sin\theta$) and viscous δ (12*d*) or resistive ε (12*e*) damping.

(ii) These values determine the parameter ν (28*a*), which specifies the asymptotic wave field, and α, β, γ ((18*a*, *b*) and (28*b*) respectively), which determine the coefficients of the hypergeometric series (41*b*).

(iii) The latter is summed using an algorithm, described elsewhere (Campos and Leitão 1988), which minimises the rounding-off error and truncates the series at a given accuracy (10^{-3} in the present case).

(iv) The altitude range corresponds to ten scale heights $0 \leq Z \leq 10$ and is covered with one hundred points in $\Delta Z = 0.1$ intervals for the computation of the variables X_0 (42*a*) and X (42*b*).

(v) The wave field (41a) is plotted, in figures 2-6, with the modulus $|U(Z)|$, i.e. the ratio of amplitude z to initial amplitude, on the LHS and the argument $\arg\{U(Z)\}$, i.e. the difference between phase at altitude z and initial phase, on the RHS.

We takes as the reference, or 'baseline', case:

$$\Omega = 2 \qquad K = 1 \tag{43a,b}$$

$$\theta = 45° \tag{43c}$$

$$\varepsilon = 0.1 = \delta \tag{43d,e}$$

and vary, in turn, each of the five parameters:

$$\Omega = 0.1, 0.5, 1.0, 2.0, 5.0 \tag{44a}$$

$$K = 0.1, 0.5, 1.0, 2.0, 5.0 \tag{44b}$$

$$\theta = 0°, 15°, 30°, 45°, 60° \tag{44c}$$

$$\delta = 0.01, 0.03, 0.10, 0.30, 0.40 \tag{44d}$$

$$\varepsilon = 0.01, 0.03, 0.10, 0.30, 1.00. \tag{44e}$$

(i) The frequency $\Omega \equiv \omega L/a = 2\pi L/\lambda$ takes values ranging from a small to a large variation of mass density $\rho(0)/\rho(\lambda) = \exp(\lambda/L) = \exp(2\pi/\Omega)$ over one reference wavelength $\lambda = 2\pi a/\omega$.

(ii) The horizontal compactness $K \equiv kL$ takes values ranging from the long wave limit $kL \ll 1$ to the ray or JWKB approximation $k^2L^2 \gg 1$, including values about unity.

(iii) The inclination of the uniform magnetic field θ takes values on either side of (43c) equal horizontal and vertical components, up to purely vertical $\theta = 0°$.

(iv) The viscous damping $\delta \equiv \eta_0\omega/a^2$ takes values from the very small $\delta \ll 1$ to the small but non-negligible $\delta^2 \ll 1$, so that dissipation can become dominant over moderate distances.

(v) The resistive damping $\varepsilon \equiv \chi_0\omega/a^2$ takes values from as small as $\varepsilon \ll 1$ to a little larger, $\varepsilon = 1$, but in all cases $\varepsilon\delta \leqslant 0.1$ so that the neglect of products of diffusivities is justified.

6. Discussion

For each of five possible combinations, leaving four parameters in (43a–e) fixed and allowing the remaining parameter to take five values (44a–e), we have a plot (shown in figures 2-6) with amplitude on the LHS and phase on the RHS as a function of altitude Z. The plots show that (i) (figure 2) as the wave frequency increases, the phase becomes larger (RHS) and the amplitude (LHS) starts to decay monotonically at higher altitude; (ii) (figure 3) as the horizontal wavenumber increases, the phase becomes larger (RHS) and the amplitude decays faster (LHS) at the higher altitude $Z \geqslant 2$; (iii) (figure 4) as the magnetic field inclination to the vertical increases, the phase also varies more rapidly (RHS) and the amplitude decays faster (LHS), though only at higher altitudes; (iv) (figures 5 and 6) increasing the viscous δ or resistive ε damping does not have much effect on phase (RHS), but causes a more pronounced 'bulge' of the amplitude at low altitude (LHS) before dissipation imposes a monotonic decay for $Z \gtrsim 3$. The most interesting feature of the preceding plots is that, although viscous and resistive dissipation cause the wave amplitude to decay at high altitude in all cases, at lower altitudes there may be a maximum, more pronounced for larger frequency $\Omega \gtrsim 2$ (figure 2), horizontal compactness $K \gtrsim 2$ (figure 3), magnetic field inclination $\theta \gtrsim 50°$ (figure 4) and viscous $\delta \gtrsim 0.3$ (figure 5) and resistive $\varepsilon \gtrsim 0.3$ (figure 6) diffusivities.

In order to explain this, note that the wave field (41*a*) is essentially a power series (41*b*) with negative exponential variable (42*b*); the simple example of a function involving the difference of two exponentials, one decaying at double the rate of the other:

$$f(Z) = \exp(-Z) - \exp(-2Z) \tag{45a}$$

$$f_{\max} = f(\log 2) = \tfrac{1}{4} \tag{45b}$$

shows that it can have a maximum, while it starts at $f(0) = 0$ and decays ultimately to zero, $f(\infty) = 0$. The reason is that, since $\exp(-2Z)$ decays much faster than $\exp(-Z)$, the difference between them initially increases, and since they both decay to zero for large Z, a maximum must exist for some intermediate moderate value of Z. A similar phenomenon, whereby the addition of electric resistance temporarily increases the magnetic field strength, is known (Moffatt 1976) in the temporal instead of spatial domain; an example is (Moffatt 1976) a magnetic field initially confined to a sphere, whose diffusion outwards is more effective for larger diffusivity, for a short time period, before dissipation causes the ultimate decay of the field. In the present problem, of dissipative magneto-atmospheric waves, the Alfvèn speed is small at low altitudes (because the mass density is relatively high) and the viscous and resistive diffusivities help the wave field to travel upward into the atmosphere, more so if it is very unsteady, i.e. for high frequency or large wavenumber. As distance increases, the Alfvèn speed (10*a*) becomes larger and the diffusivities dissipate more of the wave energy, so that the usual picture of the monotonically decaying damped wave applies. In the intermediate range, the wave amplitude may go through a maximum, as illustrated in figures 2-6.

At the maximum of the amplitude of the wave form, the gradients $\partial v/\partial z$, $\partial h/\partial z$ are determined by the phase changes and are moderate (RHS of figures 2-6); the change in amplitude near the maximum enhances the modulus of the gradients and leads to intense viscous $\dot{E}_\nu$ and resistive $\dot{E}_\sigma$ dissipation:

$$\dot{E}_\nu = \rho\nu|\partial v/\partial z|^2 \tag{46a}$$

$$\dot{E}_\sigma = (c^2/16\pi\mu\sigma)|\partial h/\partial z|^2. \tag{46b}$$

Thus we conclude that Alfvèn waves in a viscous and resistive atmosphere may be subject to intense localised dissipation by the physical process described above, which we may designate propagation-diffusive coupling; this refers to the fact that the mechanism is based on (i) the exact solution of equations describing the interaction of magnetic fields with (ii) atmospheric stratification and (iii) viscous and resistive dissipation. Other atmospheric heating mechanisms, involving hydromagnetic waves, have been proposed before, but they lack one of the essential features (i)-(iii) listed above, e.g. (*a*) the phase mixing approximation (Heyvaerts and Priest 1983) is not exact (i), i.e. is based on assumptions as to the form of the solution of the wave equation and other simplifications; (*b*) the resonance model (Hollweg 1984a, b) does not include the effects (iii) of dissipation in the calculation of the wave fields, although dissipation is necessary to heat the atmosphere; (*c*) the *RLC* analogy (Ionson 1982) does not include the effects (ii) of atmospheric stratification, since the wave speeds and damping rates are treated as effectively constant; (*d*) the variation of mean-state parameters with altitude has been considered in the ray approximation (Osterbrock 1961) but the latter does not apply to long waves. Since it is of some importance to realise the extent to which these assumptions can restrict the properties of the waves and affect the atmospheric heating mechanism, we proceed to discuss in turn the implications of the ray (*d*), *RLC* (*c*), resonance (*b*) and phase mixing (*a*) approaches.

The ray approximation requires the compactness to be large, i.e. $\omega^2 L^2/a^2 \gg 1$, which is equivalent to wavelength λ short compared to the scale height L, viz $\lambda^2 \ll 4\pi^2 L^2$. In an atmosphere, the Alfvèn speed $a(z)\to\infty$ diverges with altitude, and since the wavelength $\lambda = 2\pi a/\omega$ for a fixed frequency ω also diverges as $\lambda\to\infty$, the ray approximation breaks down. In the case (10*a*) of an isothermal atmosphere under an uniform magnetic field, the JWKB approximation $(\omega^2L^2/a_0^2)\exp(-z/L)\gg 1$ applies to high-frequency waves $\omega > a_0/L$ at low altitude $z \ll 2L\log(\omega L/a_0)$ and fails to apply to low-frequency waves $\omega < a_0/L$ at any altitude. As an example of the risks of applying the ray approximation to magneto-atmospheric waves, we give the following (others are discussed by Thomas (1982) and Campos (1983e, 1985, 1987a)): the dispersion relation $k_z n = \omega/a - mk$ implies that an Alfvèn wave would be propagating upward at low altitude, and in the case of a strictly oblique magnetic field, $m \neq 0 \neq n$, at high altitude $a\to\infty$ the wave propagates upward for $k<0$ and downward for $k>0$. Ray theory would suggest that the 'rays' would turnover, $k_z = 0$, at some intermediate altitude for $k>0$, i.e. the case $k>0$ would be singular but not the case $k<0$. However, the ray approximation does not hold at intermediate altitudes and the exact wave equation (12*a*) has a singularity at $|\rho| = 1$ in (13*a*), i.e. at altitude z_* given by (30), which is independent of k; the singularity $\rho = 1$ occurs for non-real $\bar{z}$ (31), implying that the wave field (39), calculated for (32*a*), has finite amplitude and phase. The waves exhibit finite asymptotic phase in figures 2-6 because the Alfvèn speed diverges sufficiently fast with altitude, e.g. for the case (10*a*) the asymptotic phase for propagation from altitude $z = 0$ to $z = \infty$ would be given, in the ray approximation, by

$$\Delta\phi = \int_0^\infty [\omega/A(z)]\,\mathrm{d}z = (\omega/a_0)\int_0^\infty \exp(-z/2L)\,\mathrm{d}z = 2\omega L/a_0 = 2\Omega. \tag{47}$$

Thus the ray approximation is qualitatively correct in predicting a bounded phase, but the actual value is different from (47), because (i) for non-dissipative propagating waves, the exact solution is a Hankel function of the variable in (25*a*); (ii) viscous and resistive dissipation introduces additional phase effects into the more general solution (36).

The ray approximation is implicitly used in the *RLC* analogy (Ionson 1982, 1984), because the coefficients of the dissipative Alfvèn wave equation are taken as 'lumped' constants. In this case, of constant wave speeds and damping rates, the solutions of the wave equations are complex exponentials, leading to (i) exponentially decaying amplitudes and (ii) phases which are linear functions of distance. The analytical form of the exact solution ((19), (33) or (36)), or the plots in figures 2-6, show that (i) and (ii) are not met even approximately by Alfvèn waves in a dissipative atmosphere. The problem lies not so much with the *RLC* analogy, but rather with the choice of constant parameters. A *RLC* circuit with constant coefficients represents a wave in a homogeneous medium, e.g. sound in an uniform tube or the vibrations of a string of uniform thickness. Waves in an atmosphere are similar (Campos 1986b) to the vibrations along a string of decreasing thickness, or sound in a horn of converging shape, since the mass per unit length reduces with altitude: this would lead to a *RLC* circuit with variable parameters to account for the fact that the wave experiences a varying resistance and inductance as it travels through the atmosphere. A *RLC* circuit with variable parameters no longer has the sinusoidal solutions assumed by Ionson (1982, 1984). The process of replacing the Alfvèn speed and damping rates, which vary by several orders of magnitude, e.g. over the height of the solar chromosphere, by a constant 'lumped' coefficient, eliminates some of the physics of the problem; the *RLC* circuit with constant parameters corresponds, in the present theory, to the lowest-order term ρ^{ν} in the wave field at high altitude (19), i.e. a single exponential $\sim\exp(\nu z/L)$ with monotonic decay $\mathrm{Re}(\nu)<0$, which cannot feature the maxima at intermediate altitude shown in figures 2-6. Moreover, any wave equation of second order, e.g. for Alfvèn or magnetosonic gravity waves, leads to the same *RLC* circuit, the only possible distinction being the assumed constant values of the 'lumped' coefficients. The claim (Kuperus *et al* 1981, Ionson 1985) that the *RLC* circuit analogy is independent of the particular heating mechanism is an admission that it cannot distinguish between different wave modes. The mechanism of resonant wave absorption of magneto-atmospheric waves may apply best, not to Alfvèn, but rather to magnetosonic modes (Campos and Leitão 1986, Campos 1987b) which have a critical level (Nye and Thomas 1976, Adam 1977).

The fact that the properties of Alfvèn waves are significantly affected by atmospheric stratification has been recognised for some time (Ferraro and Plumpton 1958) and has been used in theories of atmospheric heating, such as the resonance model (Hollweg 1984a, b), using three isothermal layers. The heating of an atmosphere by waves depends on the presence of dissipation processes to extract energy from the waves and deposit it on the mean state. These dissipation mechanisms, e.g. viscosity or electrical resistance, also affect the waveforms, viz amplitudes and phases. Thus it is somewhat contradictory to neglect dissipation in the calculation of the wave fields, and then to consider it in order to estimate the energy input to the atmosphere. The implication is that theories of atmospheric heating, based on non-dissipative solutions of the Alfvèn wave equations should be compared against dissipative solutions, to check whether: (i) the same physical effects occur, e.g. are the resonance or damping conditions qualitatively similar with and without dissipation? (ii) even if they are, there must be quantitative differences, e.g. the amplitude in a resonance condition is limited by damping. Both of the issues (i) and (ii) could lead to changes in the estimates of the energy deposited by the waves in the atmosphere and thus the effects of dissipation on the waveforms cannot be safely ignored.

A common claim to atmospheric wave heating mechanisms, e.g. the *RLC* analogy (Ionson 1982), the resonance model (Hollweg 1984a) and the phase mixing approximation (Heyvaerts and Priest 1983), is that the rate of heating is independent of the value of the diffusivities and is determined only by wave properties. This is the case for the absorption of a magnetosonic gravity wave at a critical level (McKenzie 1973, Campos and Leitão 1986), since the reduction of wave amplitude, and the phase change across it, determine the energy loss. These results emphasise the importance of accurate calculation of the wave fields, implying a careful consideration of all approximations made. One of the most ingenious and intriguing approaches to the study of magneto-atmospheric waves is the phase mixing approximation (Heyvaerts and Priest 1983, Nocera *et al* 1984), because it includes all essential effects, but it is debatable on at least three points: (i) it starts with the dissipative Alfvèn wave equation in the form (7), which does not apply in the case (6*a*) of non-uniform Alfvèn speed; (ii) it assumes that a solution exists in the form

$$v(x, z, t)=\int\int_{-\infty}^{+\infty} W(x, z) \exp[\mathrm{i}(k(z)x-\omega t)]\, \mathrm{d}k\, \mathrm{d}\omega \tag{48}$$

with the main z dependence in the wavenumber $(k(z)$ in the x direction, instead of taking the Fourier decomposition (8), which is formally justified; (iii) it assumes a constant dynamic diffusivity $\eta=\bar{\eta}/\rho\sim$ constant, and since the kinematic viscosity varies like the inverse square root temperature $\bar{\eta}\sim T^{-1/2}$ this implies that mass density ρ and temperature T are related by $\rho\sim T^{-1/2}$, which is a poor fit to most atmospheres, e.g. solar. We will defer a detailed examination of the phase mixing approximation (48) to another paper and address here only the crucial question for the feasibility of the heating mechanism, which relies on the phase differences between adjoining wave components of the wave, to develop into large waveform gradients, leading to intense dissipation, even by small diffusivities. Without the phase mixing 'assumption' implied in (48), it is not clear whether (*a*) the phase differences would build up until diffusion becomes effective at damping them or (*b*) the dissipation could cause wave decay before waveform shearing would become significant. The plots on the RHS of figures 2–6 support hypothesis (*b*), because there is no evidence of large phase changes for a wide range of values of wave and atmospheric parameters. The build-up of large phase differences may be the heating mechanism more suited to magnetosonic gravity waves near a critical level (Campos 1985, 1987b).

In the present work we also reach the conclusion that Alfvèn waves in a viscous and resistive atmosphere can be subject to localised dissipation and thus cause intense heating. This supports the idea that heating of atmospheres by hydromagnetic waves is physically possible; whether it does occur in specific cases, e.g. for the heating of the solar chromosphere or corona, depends on detailed modelling which is beyond the scope of the present work. One of the reasons that theories of atmospheric heating by waves are controversial may be that they involve assumptions, such as the *RLC* analogy with constant coefficients, the neglect of dissipation in the calculation of waveforms or the phase mixing or ray approximations, which are open to debate. The availability of exact solutions involving the three essential ingredients (propagation along the magnetic field, atmospheric stratification and viscous and resistive dissipation) should give more credibility to the feasibility of heating atmospheres by hydromagnetic waves. The main features of the present model are: (i) the physical mechanism for localised Alfvèn wave dissipation is an interaction of propagation and diffusion effects in space, somewhat analogous to properties of unsteady magnetic fields in time (Moffatt 1976); (ii) wave amplitude and phase can be plotted at all altitudes for a variety of combinations of frequency, wavenumber, magnetic field inclination and viscous and resistive diffusivities; (iii) the ranges of these parameters for which intense dissipation can occur can be determined, together with an indication of the relevant altitude range and maximum amplitude.

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